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An extension of Key's principle to nonlinear elasticity

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Abstract. A variational principle for finite isothermal deformations of anisotropic compressible and nearly incompressible hyperelastic materials is presented. It is equivalent to the nonlinear elastic field (Lagrangian) equations expressed in terms of the displacement field and a scalar function associated with the hydrostatic mean stress. The formulation for incompressible materials is recovered from the compressible one simply as a limit. The principle is particularly useful in the development of finite element analysis of nearly incompressible and of incompressible materials and is general in the sense that it uses a general form of constitutive equation. It can be considered as an extension of Key's principle to nonlinear elasticity. Various numerical implementations are used to illustrate the efficiency of the proposed formulation and to show the convergence behaviour for different types of elements. These numerical tests suggest that the formulation gives results which change smoothly as the material varies from compressible to incompressible.

Key words: nonlinear elasticity, near incompressibility, variational formulation, finite deformations, finite elements.

1. Introduction

It is often assumed that rubberlike materials are nearly incompressible materials. The near-incompressibility of the material can often lead to numerical difficulties [1–3] when a numerical solution, such as the finite element displacement solution, is sought. However, in the context of linear elasticity a good understanding of this phenomenon has been given by various authors, see *e.g.* [1] and [4]. In the past, many finite element models, based on penalty methods, selective-reduced integration schemes, 'approximate constraints', mixed, Lagrange multiplier, field consistent and orthogonal projection methods for linear and nonlinear (finite) elasticity have appeared in the literature, see e.g. [5–13], and good results may be obtained from some of them. A finite element solution for a nearly incompressible problem can also be obtained from the corresponding perturbed incompressible problem and an example of such a solution can be found in [14] where the method is based on the work of Spencer [15]. The boundary element method can also give good results for nearly incompressible and for incompressible linear problems (see refs. [16], [17]), but its extension to geometrically nonlinear problems is not straightforward, see *e.g.* [18]. Here, however, we concentrate on developing a mixed finite element method for nonlinear anisotropic elastic materials. Mixed methods, in the linear case, can overcome the locking problem for Poisson ratio ν equal to or close to one half, and examples may be found in [2, pp. 9–24]. An advantage of certain mixed methods, in the context of linear constraints is that they generally yield good approximations to the 'pressures' also; see the comment made by Babuska and Suri [19, p. 440].

The aim of this paper on the theoretical side is to develop a Lagrangian model, which is general in the sense that it admits a general form of strain energy function, for both compressible and incompressible materials. This formulation may then be used with commercial finite element codes (the formulation in this paper uses the pre- and post-processing of the PERFINE [20] finite element software). Numerous existing forms of the strain energy function belong to the proposed general form developed here. The work here is an extension of the recent work of Shariff [10] on isotropic elasticity. However, the proposed formulation (and the approach) here is different from that of [10] since it is motivated by suitability for future numerical computation using a modified augmented Lagrangian method, recently developed by Shariff [21].

On the computational side we indicate, in Section 5, various possible implementations of element types in the nonlinear model. We discuss the performance of only certain types of one- and two-dimensional elements. In the case of the two-dimensional element we discuss the performance of the popular, but unstable, Q1-P0 [2] element and a LBB Q2-Q1 [2, page 34] stable element on various numerical simulations. Note that the stability properties of the above two-dimensional elements depend crucially on the linear plane incompressibility constraint. Other types of elements could be used in the proposed formulation but it is beyond the scope of this paper to discuss the performances of all types of elements.

On specialising the proposed variational principle to linear (classical) elasticity, Key's principle is recovered for anisotropic materials.

2. Compressible hyperelastic materials and the incompressible limit

Following the works of Ogden [22, pp. 508–509], amongst others, we consider the modified deformation gradient tensor $\boldsymbol{F}^*$, defined in terms of the deformation gradient $\boldsymbol{F}$ as follows

$$\boldsymbol{F}^* = J^{-1/3}\boldsymbol{F}, \quad J = \det \boldsymbol{F} \tag{1}$$

so that

$$\det \boldsymbol{F}^* = 1. \tag{2}$$

In this way, $\boldsymbol{F} = J^{1/3}\boldsymbol{F}^*$ is composed of a pure dilation $J^{1/3}\boldsymbol{I}$ and an isochoric deformation $\boldsymbol{F}^*$. For an incompressible material, $J = 1$ for all deformations, so that $\boldsymbol{F}^* = \boldsymbol{F}$, and in a solution to a boundary-value problem for such a material we denote the deformation gradient by $\boldsymbol{F}_0$, where $\det \boldsymbol{F}_0 = 1$.

The strain energy function W is treated as a function of the Green strain tensor $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$, rather than of $\boldsymbol{F}$ (though the analysis can be simplified by using $\boldsymbol{F}$). Our reasons for doing so are, firstly, that $\boldsymbol{E}$ is commonly used in finite element analysis, secondly, that writing W in terms of $\boldsymbol{E}$ ensures that W is objective and, finally that in Section 4 it is convenient to use $\boldsymbol{E}$ to relate our principle to Key's principle [23].

In order to facilitate our analysis, we define a modified Green tensor

$$\boldsymbol{E}^* = \tfrac{1}{2}(\boldsymbol{F}^{*T}\boldsymbol{F}^* - \boldsymbol{I}) = J^{-2/3}\boldsymbol{E} + \tfrac{1}{2}(\boldsymbol{I}J^{-2/3} - \boldsymbol{I}) \tag{3}$$

and then express the strain energy function as

$$W(\boldsymbol{E}) = W^*(\boldsymbol{E}^*, J) \tag{4}$$

since $\boldsymbol{E}^*$ and J may be regarded as independent variables. In fact, they each are functions of $\boldsymbol{F}$, given by Equations (1) and (3), with the symmetric matrix $\boldsymbol{E}^*$ satisfying

$$\det(2\boldsymbol{E}^* + \boldsymbol{I}) = 1. \tag{5}$$

The Cauchy stress $\boldsymbol{T}$ and the second Piola–Kirchhoff stress tensor $\boldsymbol{T}^{(2)}$ are given by

$$\boldsymbol{T} = J^{-1}\boldsymbol{F}\boldsymbol{T}^{(2)}\boldsymbol{F}^T, \qquad \boldsymbol{T}^{(2)} = \frac{\partial W}{\partial \boldsymbol{E}} = J^{-2/3}\left(\frac{\partial W^*}{\partial \boldsymbol{E}^*} - P^*\boldsymbol{C}^*\right), \tag{6}$$

where

$$P^* = \tfrac{1}{3}\operatorname{tr}\left(\frac{\partial W^*}{\partial \boldsymbol{E}^*}(2\boldsymbol{E}^* + \boldsymbol{I})\right) - J\frac{\partial W^*}{\partial J} \quad \text{and} \quad \boldsymbol{C}^*(2\boldsymbol{E}^* + \boldsymbol{I}) = \boldsymbol{I}. \tag{7}$$

The derivation of (6) requires an expression for $\partial \boldsymbol{E}^*/\partial \boldsymbol{E}$, the Cartesian components of which are obtained from (3) as

$$\partial E^*_{ij}/\partial E_{rs} = J^{-2/3}\left(\delta_{ir}\delta_{js} - \tfrac{1}{3}(2E^*_{ij} + \delta_{ij})C^*_{rs}\right) \tag{8}$$

(although $E_{sr} = E_{rs}$, it is preferable in the algebraic manipulations to treat E_{sr} and E_{rs} as independent when $r \neq s$). In terms of $\boldsymbol{W}^*$, the hydrostatic part of the stress has the simple expression

$$\tfrac{1}{3}\operatorname{tr}\boldsymbol{T} = \frac{\partial \boldsymbol{W}^*}{\partial J}. \tag{9}$$

Inspection of (6), (7) and (9) shows that the reference configuration is stress-free if and only if

$$\frac{\partial W^*}{\partial J}(\boldsymbol{0}, 1) = 0, \qquad \frac{\partial W^*}{\partial \boldsymbol{E}^*}(\boldsymbol{0}, 1) = \boldsymbol{0}, \tag{10, 11}$$

while there is no loss of generality in taking the strain energy to vanish in the reference configuration:

$$W^*(\boldsymbol{0}, 1) = 0. \tag{12}$$

Moreover, one possible definition for the ground state bulk modulus is (following Ogden [22])

$$\chi = \frac{\partial^2 W^*}{\partial J^2}(\boldsymbol{0}, 1). \tag{13}$$

2.1. Incompressible materials

An incompressible material has $J = 1$, so that the Green strain tensor denoted by $\boldsymbol{E}_0 = \frac{1}{2}(\boldsymbol{F}_0^T\boldsymbol{F}_0 - \boldsymbol{I})$ must satisfy

$$\det(2\boldsymbol{E}_0 + \boldsymbol{I}) = 1. \tag{14}$$

Let $W_0(\boldsymbol{E}_0)$ denote the strain energy function. The second Piola–Kirchhoff stress is related to the deformation through

$$\boldsymbol{T}^{(2)} = \frac{\partial W_0}{\partial \boldsymbol{E}_0} - P_0 \boldsymbol{C}_0, \tag{15}$$

where $\boldsymbol{C}_0 = (2\boldsymbol{E}_0 + \boldsymbol{I})^{-1}$ and P_0 is an arbitrary scalar function. If $\boldsymbol{T}_0$ denotes the Cauchy stress then its hydrostatic part is

$$\tfrac{1}{3}\mathrm{tr}(\boldsymbol{T}_0) = \tfrac{1}{3}\,\mathrm{tr}\left(\frac{\partial W_0}{\partial \boldsymbol{E}_0}(2\boldsymbol{E}_0 + \boldsymbol{I})\right) - P_0. \tag{16}$$

As we shall see in the following, these formulae are recovered from a corresponding compressible material by taking the incompressible limit.

2.2. Strain energy functions for compressible materials and the incompressible limit

In the incompressible limit, J is entirely insensitive to the value of $p = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{T})$. In an 'almost incompressible' material, the ground-state modulus χ is large compared to all remaining ground-state moduli μ_1, μ_2,... (just a single shear modulus in the case of an isotropic material). In this case, since large pressure changes are necessary for appreciable volume changes to be produced, $J - 1$ should be regarded as a function of $\boldsymbol{E}^*$ and of ηp, where $\eta = \mu/\chi \ll 1$, with $\mu = (\mu_1^2 + \mu_2^2 + \cdots)^{1/2}$.

We introduce $\hat{W}(\boldsymbol{E}^*, p)$, a partial Legendre transform [24] of $W^*(\boldsymbol{E}^*, J)$, through

$$\hat{W}(\boldsymbol{E}^*, p) = W^*(\boldsymbol{E}^*, J) - (J-1)\frac{\partial W^*}{\partial J}, \quad p \equiv -\frac{\partial W^*}{\partial J}(\boldsymbol{E}^*, J). \tag{17}$$

Since this yields $W^*(\boldsymbol{E}^*, J) = \hat{W}(\boldsymbol{E}^*, p) - p(J-1)$, identities of the standard form arise

$$J - 1 = \frac{\partial \hat{W}}{\partial p}(\boldsymbol{E}^*, p), \qquad \frac{\partial W^*}{\partial \boldsymbol{E}^*}(\boldsymbol{E}^*, J) = \frac{\partial \hat{W}}{\partial \boldsymbol{E}^*}(\boldsymbol{E}^*, p). \tag{18}$$

While $(18)_1$ is the formal inverse of $(17)_2$, no equivalent of $(17)_2$ exists in the limit $\eta = 0$ for which $J \equiv 1$. However, for small η, we expect $J - 1$ to depend on shape changes associated with $\boldsymbol{E}^*$ and to be approximately linear in ηp so having the form

$$J = 1 - \eta J_0(\boldsymbol{E}^*) - \eta p J_1(\boldsymbol{E}^*) + \mathrm{o}(\eta).$$

Comparison with $(18)_1$ then motivates the decomposition

$$\hat{W}(\boldsymbol{E}^*, p) = W_0(\boldsymbol{E}^*) - \eta p J_0(\boldsymbol{E}^*) - \tfrac{1}{2}\eta p^2 J_1(\boldsymbol{E}^*) + \eta \hat{W}_2(\boldsymbol{E}^*, q), \tag{19}$$

where $q \equiv \eta p$ and where $\hat{W}_2$ is a smooth function of q for which

$$\hat{W}_2(\boldsymbol{E}^*, 0) = 0, \qquad \frac{\partial \hat{W}_2}{\partial q}(\boldsymbol{E}^*, 0) = 0. \tag{20}$$

This gives

$$p = \frac{1-J}{\eta J_1(\boldsymbol{E}^*)} - \frac{J_0(\boldsymbol{E}^*)}{J_1(\boldsymbol{E}^*)} + \frac{\eta}{J_1(\boldsymbol{E}^*)}\frac{\partial \hat{W}_2}{\partial q}(\boldsymbol{E}^*, \eta p), \tag{21}$$

which may, in principle, be inverted to express p in terms of the strain measures $(J-1)/\eta$ and $\boldsymbol{E}^*$. Insertion of p into $W^* = \hat{W} - p\partial\hat{W}/\partial p$ yields $W^*(\boldsymbol{E}^*, J)$ in the form

$$W^* = W_0(\boldsymbol{E}^*) + \tfrac{1}{2}\eta p^2 J_1(\boldsymbol{E}^*) + \eta\hat{W}_2(\boldsymbol{E}^*, q) - \eta q\frac{\partial \hat{W}_2}{\partial q}(\boldsymbol{E}^*, q), \tag{22}$$

while the corresponding expression for $\partial W^*/\partial \boldsymbol{E}^*$ required in (6) and (7) is

$$\frac{\partial W^*}{\partial \boldsymbol{E}^*} = \frac{\partial \hat{W}}{\partial \boldsymbol{E}^*} = \frac{\partial W_0}{\partial \boldsymbol{E}^*} - \eta p\frac{\partial J_0}{\partial \boldsymbol{E}^*} - \tfrac{1}{2}\eta p^2\frac{\partial J_1}{\partial \boldsymbol{E}^*} + \eta\frac{\partial \hat{W}_2}{\partial \boldsymbol{E}^*}(\boldsymbol{E}^*, q). \tag{23}$$

The 'incompressible limit' corresponds to $\eta \to 0$ with p finite, in which case the first term in (21) is indeterminate, so that the contribution p to P^* in (6) and (7) cannot be expressed in terms of $\boldsymbol{E}^*$ and $J-1$. This same limit gives

$$\lim_{\eta\to 0}\frac{\partial \hat{W}}{\partial \boldsymbol{E}^*} = \frac{\partial W_0}{\partial \boldsymbol{E}^*}(\boldsymbol{E}^*),$$

taking note that $J \to 1$ as $\eta \to 0$.

It may be observed that several forms of strain energy function developed previously belong to the class corresponding to (19) and (22). For example, Scott [25] uses a strain energy of the form[1]

$$W^*(\boldsymbol{E}^*, J) = \alpha(\boldsymbol{E}^*) + \beta(\boldsymbol{E}^*)(J-1) + \tfrac{1}{2}\chi(J-1)^2\gamma(\boldsymbol{E}^*) \tag{24}$$

in studying the slowness surfaces of elastic materials. This gives $p = -\beta(\boldsymbol{E}^*) - \chi\,(J-1)\gamma(\boldsymbol{E}^*)$ which leads to

$$\hat{W}(\boldsymbol{E}^*, p) = \alpha(\boldsymbol{E}^*) - \frac{\eta}{2\mu}\frac{(p+\beta(\boldsymbol{E}^*))^2}{\gamma(\boldsymbol{E}^*)}. \tag{25}$$

2.3. Specialization to isotropic elasticity

For an isotropic hyperelastic solid, W can be expressed as a symmetric function of the principal stretches λ_i $(i = 1, 2, 3)$, so that we consider a class of strain energy functions W^* associated with $\hat{W}(\boldsymbol{E}^*, p)$ of the form

$$\begin{aligned}\hat{W}(\boldsymbol{E}^*, p) \;=\; & F_0(\lambda_1^*, \lambda_2^*, \lambda_3^*) - \eta p f_0(\lambda_1^*, \lambda_2^*, \lambda_3^*)\\ & -\tfrac{1}{2}\eta p^2 f_1(\lambda_1^*, \lambda_2^*, \lambda_3^*) + \eta f_2(\lambda_1^*, \lambda_2^*, \lambda_3^*, \eta p).\end{aligned} \tag{26}$$

[1] We note that Scott proposed a slightly more general form of the strain energy function, *i.e.* $W^*(\boldsymbol{E}^*, J) = \alpha(\boldsymbol{E}^*) + \beta(\boldsymbol{E}^*)(J-1) + \frac{1}{2}(J-1)^2\,\vartheta(\boldsymbol{E}^*)$ with the property that $\alpha(\boldsymbol{E}^*)$ and $\beta(\boldsymbol{E}^*)$ are finite but do not become indefinitely large as $\vartheta(\boldsymbol{E}^*) \to \infty$.

Here, each of F_0, f_0, f_1 and f_2 is symmetric in the three arguments $\lambda_i^* \equiv J^{-1/3}\lambda_i$, which must satisfy $\lambda_1^*\lambda_2^*\lambda_3^* = 1$ and $\lambda_1\lambda_2\lambda_3 = J$. Alternatively, we may replace the arguments λ_1^*, λ_2^* and λ_3^* in (26) by any two quantities symmetric in all three, for example

$$I_1^* \equiv \lambda_1^{*2} + \lambda_2^{*2} + \lambda_3^{*2} \quad \text{and} \quad I_2^* \equiv \lambda_1^{*-2} + \lambda_2^{*-2} + \lambda_3^{*-2}. \tag{27}$$

In either case, we take $f_2(\lambda_1^*, \lambda_2^*, \lambda_3^*, q)$ to be a function of $q = \eta p$, having derivatives of O(1).

Some examples of W^* which appear in the literature are given in Appendix B, together with associated expressions for p.

3. Variational principle

The basic equations of nonlinear elasticity may be summarised as follows. Consider an elastic body occupying the region B_0 in some stress-free configuration. A point in B_0 is identified by its position vector $\boldsymbol{X}$ relative to some origin. Under deformation this point moves to a new position $\boldsymbol{x}(\boldsymbol{X})$. The displacement vector $\boldsymbol{u}$ is given by $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$.

The equations of equilibrium are

$$\operatorname{Div} \boldsymbol{S} + \rho_0 \boldsymbol{b} = \boldsymbol{0}, \quad \boldsymbol{X} \in B_0, \tag{28}$$

$$\boldsymbol{S} = \partial W/\partial \boldsymbol{D}, \qquad \boldsymbol{D} = \operatorname{Grad} \boldsymbol{u} = \boldsymbol{F} - \boldsymbol{I},$$

where ρ_0 is the mass density (per unit undeformed volume), Div and Grad denote the divergence and gradient operators relative to B_0, $\boldsymbol{S}^T$ is the first Piola–Kirchhoff stress and $\boldsymbol{b}$, the body force, is expressible as

$$\boldsymbol{b} = -\operatorname{grad} \Phi, \tag{29}$$

where Φ is a scalar function of $\boldsymbol{x}$ and grad refers to the gradient operation with respect to $\boldsymbol{x}$.

Boundary conditions of a fairly general form are considered. Let $(\boldsymbol{g}^1(\boldsymbol{X}), \boldsymbol{g}^2(\boldsymbol{X}), \boldsymbol{g}^3(\boldsymbol{X}))$ be an orthonormal set of vectors and ∂B_0 be the boundary of B_0. At each point of ∂B_0, m components of displacement are specified (m = 0, 1, 2, 3) through

$$\boldsymbol{u}.\boldsymbol{g}_i = \zeta_i(\boldsymbol{X}), \quad \boldsymbol{X} \in \partial B_0^{/i}, \tag{30}$$

where $\boldsymbol{g}_i$ is the reciprocal (or dual) basis for $\boldsymbol{g}^i$, while exactly $3 - m$ components of traction are related to the displacement and displacement gradient through

$$\boldsymbol{S}^T \boldsymbol{N}.\boldsymbol{g}^j = \sigma^j(\boldsymbol{X}, \boldsymbol{u}, \boldsymbol{D}), \quad \boldsymbol{X} \in \partial B_0^j, \tag{31}$$

where i, j can take the values of 1, 2 or 3, with

$$\partial B_0^{/i} \cap \partial B_0^i = \emptyset, \qquad \partial B_0^{/i} \cup \partial B_0^i = \partial B_0 \tag{32}$$

with no summation over i in (32) and with $\boldsymbol{N}$ the unit outward normal to ∂B_0. The above boundary conditions have been used in *e.g.* Refs. [8] and [26]. The use of the theorem of minimum potential energy (in the context of linear elasticity) in finite element displacement models generates solutions in which accuracy is adversely affected as ν approaches the critical value 0·5 [27]. The inaccuracies are associated with the fact that for $\nu = 0{\cdot}5$, the usual displacement formulation is no longer valid [27]. To avoid this difficulty, Key [23], for example, discretized the infinitesimal hydrostatic mean stress, $\sigma_{ii}/3$ ($\equiv -p$, say) independently of the displacement field and determined it from the linear volumetric dilatation by the relations (see Appendix A for the notations below)

$$e_{kk} = (-3p - C_{kkij}e^*_{ij})/3\chi. \tag{33}$$

Within the strain energy function, e_{kk} is replaced by the right hand side of Equation (33) and relation (33) is also introduced into the functional via a Lagrange multiplier. A similar principle (but a different method) is applied to nonlinear elasticity. Using $-p$ again to denote the hydrostatic mean stress $\mathrm{tr}(\boldsymbol{T})/3$, we have from (18) and (19)

$$J = 1 + \frac{\partial \hat{W}}{\partial p} = 1 - \eta J_0(\boldsymbol{E}^*) - \eta p J_1(\boldsymbol{E}^*) + \eta^2 \frac{\partial \hat{W}_2}{\partial q}(\boldsymbol{E}^*, q). \tag{34}$$

Provided that the right-hand side of (34) is monotonic and continuous in q at each $\boldsymbol{E}^*$, Equation (34) may be inverted uniquely to give p in terms of $\boldsymbol{E}^*$ and $(J-1)/\eta$. Then, the strain energy function $W^*(\boldsymbol{E}^*, J)$ associated to (19) is found through

$$W^* = \hat{W}(\boldsymbol{E}^*, p) - (J-1)p, \qquad p = P(\boldsymbol{E}^*, \eta^{-1}(J-1)).$$

Equations (28) and the associated boundary conditions (30) and (31) result from the vanishing of the first variation of the functional

$$\Pi(\boldsymbol{u}) = \int_{B_0} \{W(\boldsymbol{E}) + \rho_0\Phi(\boldsymbol{X}+\boldsymbol{u}) - \Psi\}\,\mathrm{d}V, \tag{35}$$

where $\boldsymbol{u}(\boldsymbol{X})$ is any displacement field satisfying the essential boundary condition (30), provided only that the scalar function $\Psi = \Psi(\boldsymbol{X}, \boldsymbol{u}, \boldsymbol{D})$ satisfies Lagrange equations of the form

$$\mathrm{Div}\left(\frac{\partial \Psi}{\partial \boldsymbol{D}}\right) - \frac{\partial \Psi}{\partial \boldsymbol{u}} = \boldsymbol{0}, \quad \boldsymbol{X} \in B_0 \tag{36}$$

and has boundary values which satisfy

$$\left(\frac{\partial \Psi}{\partial \boldsymbol{D}}\right)^T \boldsymbol{N}.\boldsymbol{g}^j = \sigma^j(\boldsymbol{X}, \boldsymbol{u}, \boldsymbol{D}), \quad \boldsymbol{X} \in \partial B_0^j. \tag{37}$$

Indeed, vanishing of the first variation of (35) yields

$$\begin{aligned} 0 \;=\; \delta\Pi &= \int_{B_0} \left\{\mathrm{tr}\left(\frac{\partial W}{\partial \boldsymbol{D}}\delta\boldsymbol{D}\right) - \rho_0\boldsymbol{b}.\delta\boldsymbol{u} - \frac{\partial \Psi}{\partial \boldsymbol{u}}.\delta\boldsymbol{u} - \mathrm{tr}\left(\frac{\partial \Psi}{\partial \boldsymbol{D}}\delta\boldsymbol{D}\right)\right\}\mathrm{d}V \\ &= \int_{B_0}\left\{-\mathrm{Div}\left(\frac{\partial W}{\partial \boldsymbol{D}}\right) - \rho_0\boldsymbol{b} + \mathrm{Div}\left(\frac{\partial \Psi}{\partial \boldsymbol{D}}\right) - \frac{\partial \Psi}{\partial \boldsymbol{u}}\right\}.\delta\boldsymbol{u}\,\mathrm{d}V \\ &\quad + \int_{B_0}\left\{\boldsymbol{N}^T\frac{\partial W}{\partial \boldsymbol{D}} - \boldsymbol{N}^T\frac{\partial \Psi}{\partial \boldsymbol{D}}\right\}.\delta\boldsymbol{u}\,\mathrm{d}A. \end{aligned}$$

This formulation requires identification of a solution $\Psi(\boldsymbol{X}, \boldsymbol{u}, \boldsymbol{D})$ to (36) in B_0 satisfying the boundary conditions (37) on the three portions of boundary ∂B_0^j over which traction components are specified as in (31), but requires no restrictions on Ψ on portions of ∂B_0 where $\delta \boldsymbol{u} = \boldsymbol{0}$. In the special case for which σ^j depends on $\boldsymbol{X}$ and $\boldsymbol{u}$, it is possible to replace (35) by

$$\Pi(\boldsymbol{u}) = \int_{B_0} \{W(\boldsymbol{E}) + \rho_0 \Phi\} \, \mathrm{d}V - \int_{\partial B_0^\sigma} \psi \, \mathrm{d}A, \tag{38}$$

where $\psi(\boldsymbol{X}, \boldsymbol{u})$ is a scalar function such that

$$\frac{\partial \psi}{\partial \boldsymbol{u}} . \boldsymbol{g}^j = \sigma^j(\boldsymbol{X}, \boldsymbol{u}), \quad \boldsymbol{X} \in \partial B_0^j \tag{39}$$

and

$$\partial B_0^\sigma = \bigcup_{j=1}^{3} \partial B_0^j .$$

In Equation (35), the term $W(\boldsymbol{E})$ may be expressed in terms of $\boldsymbol{E}^*$ and J as

$$W(\boldsymbol{E}) = W^*(\boldsymbol{E}^*, J) = \hat{W}(\boldsymbol{E}^*, p) - p \frac{\partial \hat{W}}{\partial p}, \quad p \equiv -\frac{\partial W^*}{\partial J}(\boldsymbol{E}^*, J),$$

where J and $\boldsymbol{E}^*$ are themselves defined in terms of $\boldsymbol{u}$ through (1) and (3), with $\boldsymbol{F} = \boldsymbol{I} + \mathrm{Grad}\, \boldsymbol{u}$. Alternatively, by defining $J_p(\boldsymbol{E}^*, p)$ as the solution to

$$\frac{\partial W^*}{\partial J}(\boldsymbol{E}^*, J_p) = -p, \tag{40}$$

while still regarding J and $\boldsymbol{E}^*$ as derived from $\boldsymbol{u}$, we replace the functional in (35) by one defined in terms of the independent fields $\boldsymbol{u}$ and p as

$$\Pi^*(\boldsymbol{u}, p) = \int_{B_0} \{W^*(\boldsymbol{E}^*, J_p) + (J_p - J)p + \rho_0 \Phi - \Psi\} \, \mathrm{d}V. \tag{41}$$

This is appropriate when σ^j depends on $\boldsymbol{X}$, $\boldsymbol{u}$ and $\boldsymbol{D}$ as in (37). Similarly, if σ^j depends only on $\boldsymbol{X}$ and $\boldsymbol{u}$, the functional in (38) is re-expressed as

$$\Pi^*(\boldsymbol{u}, p) = \int_{B_0} \{W^*(\boldsymbol{E}^*, J_p) + (J_p - J)p + \rho_0 \Phi\} \, \mathrm{d}V - \int_{\partial B_0^\sigma} \psi \, \mathrm{d}A. \tag{42}$$

Recalling that in (41) and (42) both $\boldsymbol{E}^*$ and J are computed from $\boldsymbol{F} \equiv \boldsymbol{I} + \mathrm{Grad}\, \boldsymbol{u}$, while $J_p = J_p(\boldsymbol{E}^*, p)$, defined through (40), depends also on p, we find that whenever $\boldsymbol{u}$ belongs to a set of kinematically admissible deformations which are suitably smooth and which satisfy the essential boundary condition (30) while p is also sufficiently smooth, the vanishing of the first variation of Π^* gives

$$\begin{aligned} 0 = \delta \Pi^* = & \int_{B_0} \left\{ \mathrm{tr}\left(\frac{\partial W^*}{\partial \boldsymbol{E}^*} \delta \boldsymbol{E}^* \right) + \frac{\partial W^*}{\partial J} \delta J_p + (J_p - J) \delta p \right. \\ & \left. + p \delta J_p - p \delta J - \rho_0 \boldsymbol{b} . \delta \boldsymbol{u} \right\} \mathrm{d}V - \int_{\partial B_0^\sigma} \sigma^j \delta u_j \, \mathrm{d}A. \end{aligned} \tag{43}$$

Here $\sigma^j \delta u_j$ is summed over the region where the components of the surface traction are prescribed, while δu_j are the components of $\delta\boldsymbol{u}$ written in terms of the basis vectors $\boldsymbol{g}^j$. Equation (43) yields the boundary conditions (37) together with the field Equations (28), in which $\boldsymbol{S}$ is now expressed as

$$\boldsymbol{S} = J^{-2/3}\left(\frac{\partial W^*}{\partial \boldsymbol{E}^*} - \tfrac{1}{3}\mathrm{tr}\left[(2\boldsymbol{E}^* + \boldsymbol{I})\frac{\partial W^*}{\partial \boldsymbol{E}^*})\right]\boldsymbol{C}^* - Jp\boldsymbol{C}^*\right)\boldsymbol{F}^T = \boldsymbol{T}^{(2)}\boldsymbol{F}^T. \tag{44}$$

It also yields the condition $J(= \det \boldsymbol{F}) = J_p(\boldsymbol{E}^*, p)$. This validates the use of either of the functionals (41) or (42) for determining displacements and stresses.

In the incompressible limit $\eta \to 0$, the stress distribution corresponds to $\boldsymbol{T}^{(2)} \to (\partial W_0/\partial \boldsymbol{E}^*) - P^*\boldsymbol{C}^*$, with $J \to 1$ and with $P^* \to \frac{1}{3}\mathrm{tr}\{(\partial W^*/\partial \boldsymbol{E}^*)(2\boldsymbol{E}^* + \boldsymbol{I})\} - p$. In the finite element method, Equation (43) is solved for an appropriate class of $\boldsymbol{u}$ and p. It is clear that the formulation is valid for both compressible and incompressible materials, just like the similar analysis of Refs. [23] and [28] for linear elasticity. In the formulation, η appears explicitly. In practice, η will become numerically negligible while χ remains finite, at a value dependent on the computing machine.

For isotropic materials, the variational principles may be written in terms of λ_1^* and λ_2^* (or I_1^* and I_2^*) by replacing $\boldsymbol{E}^*$ in (41) or (42) much as in subsection 2.3. As in the general anisotropic case, the resulting Euler equations are equivalent to the equations of the corresponding boundary value problem.

4. Connection with Key's principle

On specializing our principles to the classical linear theory of elasticity we obtain the functionals

$$\Pi^*(\boldsymbol{u}, p) = \int_{B_0} \{-pe_{kk} - (p + \tfrac{1}{3}S_{rrij}e_{ij}^*)^2/2\chi + \tfrac{1}{2}S_{ijkl}e_{ij}^*e_{kl}^* - \rho_0\boldsymbol{b}.\boldsymbol{u}\}\,\mathrm{d}V - \int_{\partial B_0^\sigma} \sigma^j u_j\,\mathrm{d}A \tag{45}$$

for anisotropic materials [See Appendix A for the above notations] and

$$\Pi^* = \int_{B_0} \{\mu e_{ij}^* e_{ij}^* - pe_{kk} - \tfrac{1}{2}\chi^{-1}p^2 - \rho_0\boldsymbol{b}.\boldsymbol{u}\}\,\mathrm{d}V - \int_{\partial B_0^\sigma} \sigma^j u_j\,\mathrm{d}A \tag{46}$$

for isotropic materials.

The above principles are similar to that of Key's principle [23], in the case of an isothermal deformation.

5. Numerical examples

The formulation above is currently being implemented in the PERFINE [20] finite element software. Here, to give confidence in the formulation, numerical solution of a one-dimensional problem is outlined, using a Newton–Raphson method with incremental loading to solve the

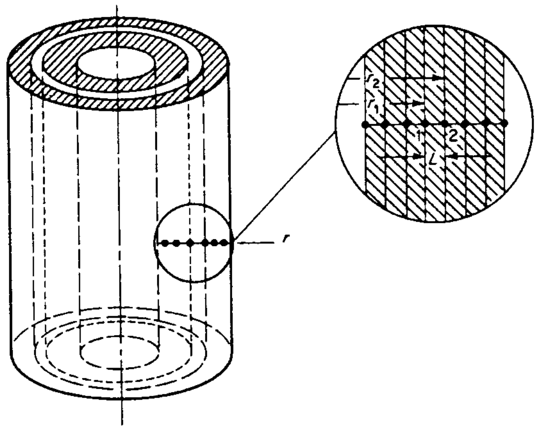

Figure 1. Finite-element representations of an infinite undeformed cylinder.

discrete equations. The formulation is also tested numerically on a few two-dimensional problems, and will be extended in the future using a modified augmented Lagrangian method recently developed by Shariff [21]. The behaviour of Q1-P0 and the LBB Q2-Q1 two-dimensional elements is investigated using a neo-Hookean elastic material with strain energy function given by Blatz [29] (See Appendix B)

$$W^* = \frac{E}{4(1+\nu)}\{(\lambda_1^{*2} + \lambda_2^{*2} + \lambda_3^{*2})J^{2/3} - 3 - 2(J-1)\} + \frac{3\chi(J - 1 - \log J)}{2(1+\nu)}, \qquad (47)$$

where E is the ground-state Young's modulus. The Q1-P0 rectangular element consists of bilinear functions (4 nodes) approximating the displacement, with the pressure taken as constant over the element. The Q2-Q1 rectangular element consists of quadratic functions (8 nodes) approximating the displacement and bilinear functions approximating the pressure. An element is LBB stable if it satisfies the div-stability [2] condition, otherwise it is unstable. The two-dimensional problems are computed on an INMOS Transputer with 2MB of RAM. Double-precision arithmetic is used and the tolerance for the residual of Equation (43) is set at $10^{-6}\times$ relevant factor.

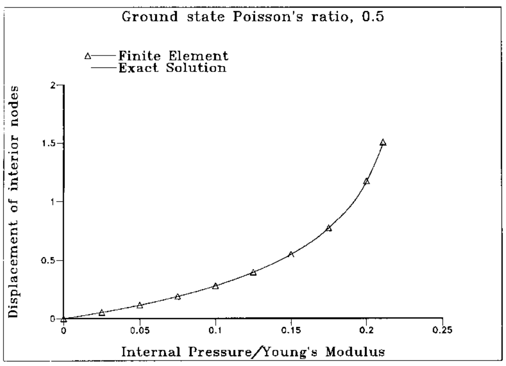

Figure 2. Comparison of exact displacement with finite element displacement for incompressible material.

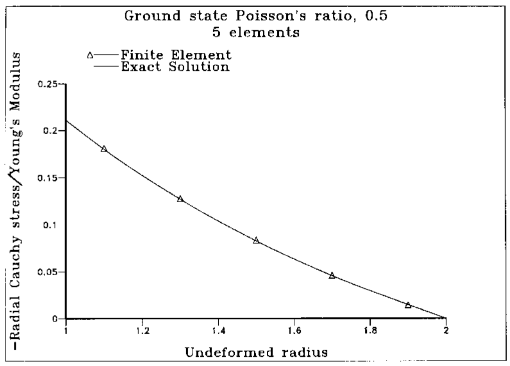

Figure 3. Comparison of exact stress with finite element stress for incompressible material using 5 elements.

5.1. ONE-DIMENSIONAL TEST PROBLEM: INFLATING AN INFINITELY LONG THICK-WALLED TUBE WITH AN INTERNAL PRESSURE

This example is a simple test problem for which the results of the proposed formulation can be compared with *exact* and previous finite element calculations. Exact and finite element solutions for incompressible material can be found in Green and Zerna [31, pp. 88–92] and Oden [32, pp. 321–331], respectively. The internal and external radii of the tube are taken as 1 and 2 units, while the strain energy function (47) is used. Internal pressures as high as $0.211E$ are considered, corresponding to strains of order 150%, so that the behaviour falls outside that capable of being predicted by classical theory. For simplicity, results shown in Figures 2-4 use only 5 linear one-dimensional elements spaced equally, as shown in Figure 1. Since p is a measure of stress, it is taken as constant within each element, with value denoted by p_e. To permit the reader to check on the results, details of the finite element equations are now given. Let u^1, u^2 and R_1, R_2 be respectively the nodal radial displacements and radial coordinates of an element. We then have the approximations

$$I_1^* = \lambda_1^{*2} + \lambda_2^{*2} + \lambda_3^{*2} = (1 + \Delta^2 + \omega^2)/(\omega\Delta)^{2/3},$$

where

$$\Delta = 1 + (u^2 - u^1)/(R_2 - R_1), \qquad \omega = 1 + \bar{u}/\bar{R},$$

$$\bar{u} = (u^1 + u^2)/2 \quad \text{and} \quad \bar{R} = (R_1 + R_2)/2.$$

Within each element, we determine the value of J_p from I_1^* and p_e by solving numerically the equation

$$J_p = 1 + \frac{2(1+\nu)}{3\chi}(-p_e J_p - \mu(\tfrac{1}{3} I_1^* J_p^{2/3} - J_p)). \tag{48}$$

The nonlinear element stiffness relations then are

$$\pi(R_2^2 - R_1^2)\left(\frac{(-1)^N A}{L} + \frac{B}{2\bar{R}}\right) = f^N, \quad N = 1, 2,$$

together with

$$\omega\Delta = J_p,$$

where $L = R_2 - R_1$, $J = \omega\Delta$, $\mu = E/(2(1+\nu))$, where f^N are the elemental loads and where

$$A = \mu J_p^{2/3}\left(\frac{\Delta}{J^{2/3}} - \frac{\omega I_1^*}{3J}\right) + p\omega, \qquad B = \mu J_p^{2/3}\left(\frac{\omega}{J^{2/3}} - \frac{\Delta I_1^*}{3J}\right) + p\Delta.$$

The global nonlinear stiffness equations are solved using a Newton–Raphson method together with 4 steps of incremental loading. From the graphs illustrated in Figures 2–3 it can be seen that our solution agrees well with the theoretical solution [30]. Note that the theoretical (exact) solution requires numerical determination of a parameter c from the equation

$$\text{internal pressure} = \tfrac{1}{6}E\left\{\log(1+c) + \frac{c}{1+c} - \log(4+c) - \frac{c}{4+c} + \log\, 4\right\},$$

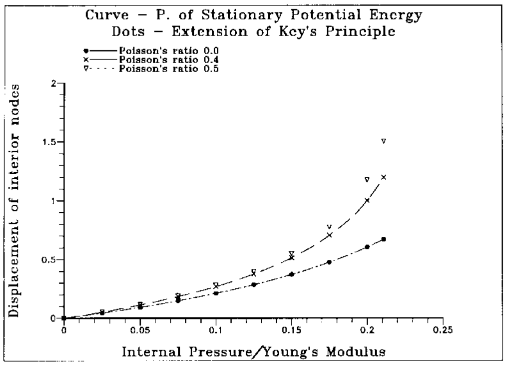

Figure 4. Comparison of our Principle with the Principle of Stationary Potential Energy.

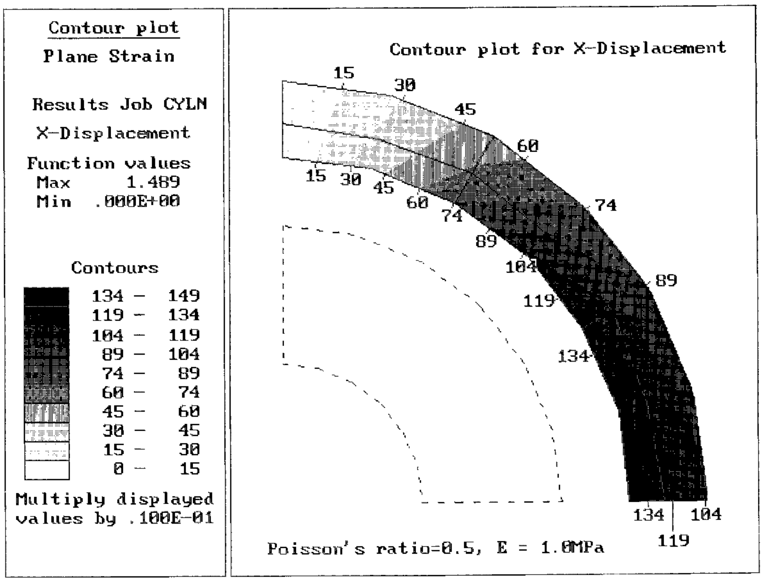

Figure 5. Plane strain problem of inflating an infinitely long circular tube by an internal pressure 0·211E. Deformed and undeformed configuration. Q2-Q1 elements.

and the displacement u at $R = 1$ (say) is given by $u = \sqrt{1 + c} - 1$.

The number of iterations required for $\nu = 0{\cdot}499$, $0{\cdot}4999$, $0{\cdot}49999$, $0{\cdot}499999$ and $0{\cdot}5$ are identical. For smaller ν, the number of iterations remains about the same. No numerical difficulties were encountered for $0 \leqslant \nu \leqslant 0{\cdot}5$. When our solution is compared (see Figure 4), for smaller values of ν (*i.e.* $\nu = 0$ and $\nu = 0{\cdot}3$), with a solution obtained using the standard principle of stationary potential energy, the solutions agree completely to four significant figures.

We also compare the one-dimensional results to the equivalent two-dimensional plane strain results using six Q2-Q1 elements (see Figure 5). We note that the results are similar to those using 5 one-dimensional elements and are indicated in Figure 5.

5.2. Two-dimensional problems

We investigate the performances of the Q1-P0, the eight-noded 'serendipity' Q2-Q1 and the Q1 (based on the standard displacement model) elements on plane strain and axisymmetric problems of bonded elastic mounts. The problems concern flat deformable blocks bonded between two parallel rigid end-plates. The plane strain problem concerns a rectangular strip of infinite breadth whereas the axisymmetric problem concerns a circular disc. The mounts are subjected to tension and compression. The strain energy used in the calculation is of the form given in Equation (47). The solution for the Q1 element (associated with the standard displacement model) is obtained from the parallel element software developed by Shariff [32], where a nonlinear Jacobi preconditioned conjugate gradient method is used to solve the system of nonlinear equations. In the case of the Q1-P0 and Q2-Q1 elements the system of nonlinear equations is solved via the Newton–Raphson method. In most calculations, ten increments are used and convergence generally occurs after two iterations per increment.

5.3. Plane-strain tension

Both faces of the rigid block are displaced to give symmetry, so that a mesh need cover only a quarter of the block. This is an example of an inhomogeneous test problem. Figures 6 and 7 depict the deformations and the Cauchy stress for various compressibility moduli. The figures indicate severe stress inhomogeneities near the bonded edge for both Q1-P0 and Q2-Q1 elements. Using a different formulation, Miehe [11] observes similar behaviour for the Kirchhoff stress. In Figure 8 the convergence as the number of elements is increased is depicted; it shows the scale force F/E for the axial stretch $\bar{\lambda}_2 = 3$, plotted against the number of mesh elements in a quarter of the block. The force is calculated using a variational equation similar to Equation (43) not from the stress calculated at the bonded surface, which is discontinuous. F is an integral quantity, which characterizes the whole system. In the compressible case, the results for the axial force are nearly independent of the number of elements. However, in the case of nearly incompressible and incompressible materials, the Q2-Q1 element performed better in the sense that F/E for the Q2-Q1 element approaches a limit using fewer elements. We should, however, note that the Q2-Q1 element has more degrees of freedom than either the Q1-P0 or Q1 elements. The Q1 element solution exhibited an extreme locking phenomenon for near-incompressibility. Corresponding results are not plotted. For moderate values of $\bar{\lambda}_2 - 1$, Shariff [33] has shown analytically that the value of F/E obtained by a finite element displacement model is an upper bound for the actual force; Figure 8 seems to indicate a similar behaviour, although no rigorous analyses have been made to validate this behaviour, using the proposed formulation. The formulation proposed by Miehe [11] also indicates such behaviour. Figure 8 indicates that the tensile force for a compressible material exceeds the tensile force for an incompressible (or nearly incompressible) material. Initially this result was not expected; Miehe [11] had shown otherwise, using Ogden's [22] material. This seemingly odd result is due to the Blatz material, not to the finite element formulation, as explained by Shariff [10].

5.4. Plane strain compression and axisymmetric deformation

The behaviour of the finite element solutions for plane strain compression and axisymmetric deformation is similar to that of plane strain tension. Hence we shall not depict the results.

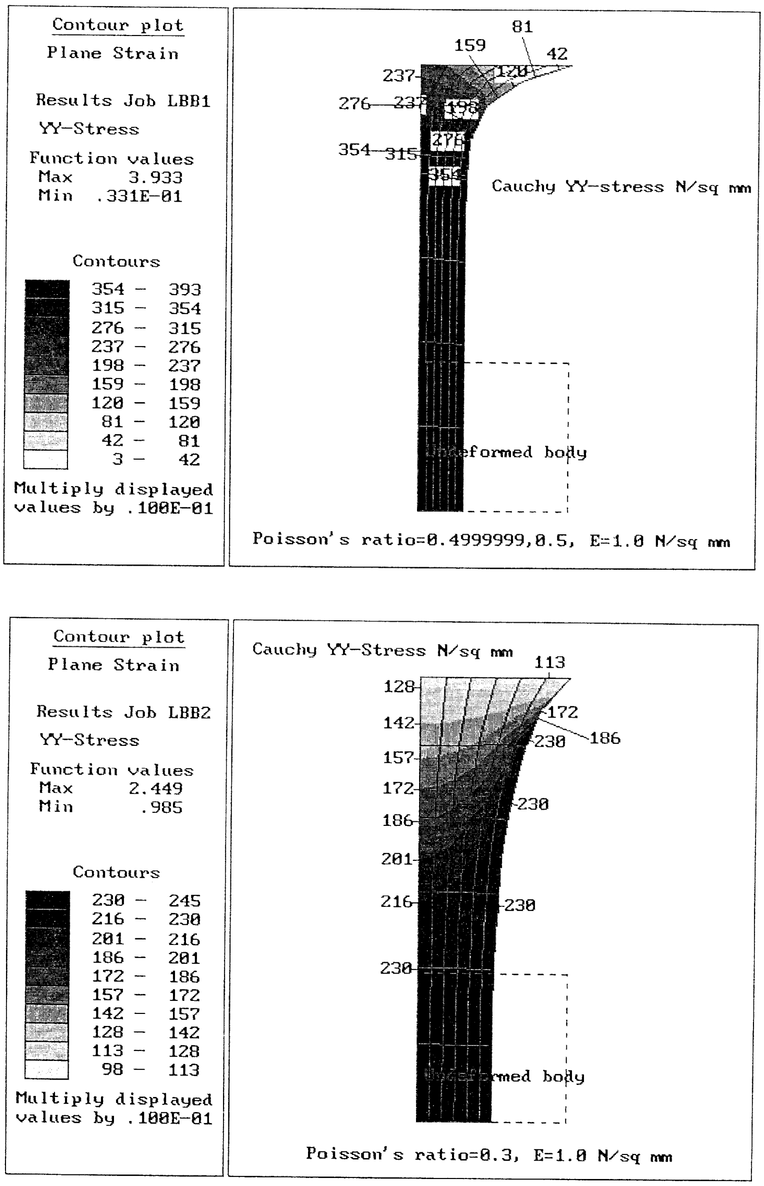

Figure 6. Plane strain tension test: Q2-Q1 element. Deformed configuration and a component of the Cauchy stress T_{22}, at axial stretch $\bar{\lambda}_2 = 3$ for various values of Poisson's ratio.

However, for thin blocks the volume change for nearly incompressible material under compression can be significant. Volume changes per unit undeformed volume are calculated and it is found that the volume change increases with s (the ratio of one loaded area to the corresponding force-free undeformed area), as expected [33]. We also observed that for the case of axisymmetric deformation, for $s = 2{\cdot}39$, $E = 17{\cdot}5\,\text{kg}$ and $\nu = 0{\cdot}49971$ (these values are those obtained by Gent and Lindley [34] for soft gum rubber vulcanizate in their experiment), the volume change for both types of element is $-0{\cdot}5584 \times 10^{-2}$ and this is an order of magnitude higher than $\mu/\chi = 0{\cdot}5795 \times 10^{-3}$; this indicates that the proposed formulation is also

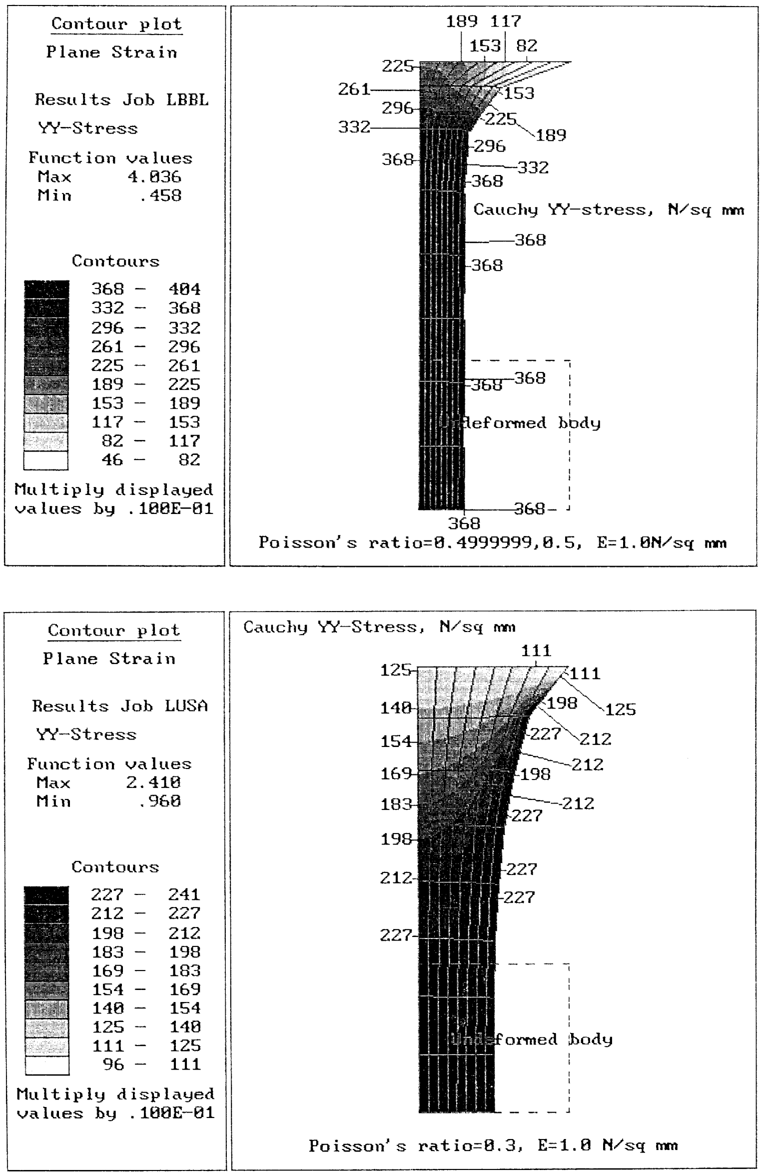

Figure 7. Plane strain tension test: Q1-P0 element. Deformed configuration and a component of the Cauchy stress T_{22}, at axial stretch $\bar{\lambda}_2 = 3$ for various values of Poisson's ratio.

valid for moderate volume changes unlike some previous formulations where an assumption of small volume change (of order μ/χ) is required for nearly incompressible material.

5.5. Some remarks

For all the above types of elements singular stress fields are mildly detected near the bonded edges although they are not clearly shown in Figures 6 and 7. The magnitude of the gradients of the stresses increases sharply just adjacent to the bonded edges. However, when the number of elements is increased near the bonded edges singular stress fields are more significantly displayed there. The use of special singular elements could give a better picture of the stress

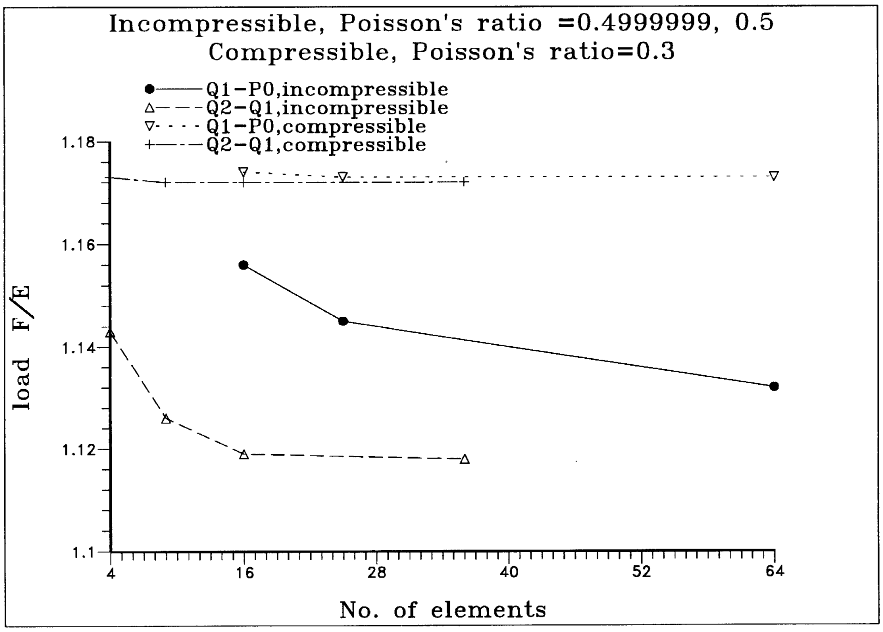

Figure 8. Plane strain tension test: Convergence of different elements.

fields near the bonded edges. We note that we found no evidence of oscillatory stress behaviour using the above types of elements and this is clearly indicated in Figures 6 and 7. We also note that in the present paper we have used the Newton–Raphson (with incremental loading) method to obtain the solutions. This method is not suitable for calculating solutions for unstable problems; an example of an effective method for unstable problems is a modified Riks method [36]. Nevertheless, we found that, for example, in axisymmetric compression, the Q1-P0 elements deformed unreasonably [10] near the bonded edges for $L/2H = 2{\cdot}39$ and $L/2H = 2{\cdot}15$ at 15% (and higher) and 17·5 % (and higher) compressions, respectively. This could indicate the possibility of surface (or global) instabilities for the compression problem.

6. Conclusion

Variational principles have been developed which permit the description of the behaviour of nonlinear deformations of compressible and incompressible elastic solids (which need not be isotropic). The principles are general in the sense that they permit use of a general form of strain energy function. The numerical examples using various types of elements demonstrate that accurate results can be obtained for the analysis of compressible, slightly compressible and incompressible materials. The principle may be regarded as an extension of Key's principle to nonlinear elasticity.

Acknowledgement

To Tony Spencer on his 70th birthday – a most exceptional teacher and colleague.

Appendix A

The linearisation of Equations (6) and (7) should be consistent with the linear elastic relations

$$\sigma_{ij} = C_{ijkl}e_{kl} = C_{ijkl}e^*_{kl} + \tfrac{1}{3}C_{ijkk}e_{rr}, \tag{A1}$$

in which $e^*_{ij} = e_{ij} - \frac{1}{3}\, e_{kk}\delta_{ij}$ are the components of deviatoric strain, while σ_{ij} and e_{ij} are the cartesian coordinates of Cauchy stress and infinitesimal strain respectively. Additionally, C_{ijkl} are the classical elastic coefficients for anisotropic hyperelastic materials and δ_{ij} is the Kronecker delta. Within linear theory, we have the approximations

$$E^*_{ij} \approx e^*_{ij}, \qquad J - 1 \approx e_{rr}\,. \tag{A2}$$

Since (A1) shows that the hydrostatic mean stress is given by

$$-p = \tfrac{1}{3}\sigma_{ii} = \tfrac{1}{3}C_{iikl}e^*_{kl} + \tfrac{1}{9}C_{iirr}e_{kk}, \tag{A3}$$

it is clear from (A3) that in the incompressible limit ($e_{kk} \to 0$) C_{iirr} becomes infinite in such a way that the product $C_{iirr}e_{kk}$ remains finite, but indeterminate. Let

$$\bar{\chi} = \tfrac{1}{9}C_{iirr}. \tag{A4}$$

The behaviour as $\bar{\chi} \to \infty$ is clarified by considering a class of elastic coefficients that can be defined as follows

$$C_{ijkl} = S_{ijkl} + \bar{\chi}\delta_{ij}\delta_{kl}, \tag{A5}$$

where S_{ijkl} do not become indefinitely large as $\bar{\chi} \to \infty$. From (A4) and (A5) it is clear that $S_{iirr} = 0$ and that (A3) becomes

$$\tfrac{1}{3}\sigma_{ii} = \tfrac{1}{3}S_{iikl}e^*_{kl} + \bar{\chi}e_{rr}, \tag{A6}$$

while (A1) becomes

$$\sigma_{ij} = S_{ijkl}e^*_{kl} + (\tfrac{1}{3}S_{ijkk} + \chi\delta_{ij})e_{rr}. \tag{A7}$$

In linearisation of (6) and (7), we define

$$M_{ijkl} = \frac{\partial^2 W^*}{\partial E^*_{ij}\partial E^*_{kl}}(\mathbf{0}, 1) = M_{klij}, \qquad N_{ij} = \frac{\partial^2 W^*}{\partial E^*_{ij}\partial J}(\mathbf{0}, 1), \qquad \chi = \frac{\partial W^*}{\partial J^2}(\mathbf{0}, 1)$$

so yielding

$$P^* \approx -(\chi - \tfrac{1}{3}N_{pp})e_{rr} + (\tfrac{1}{3}M_{ppkl} - N_{kl})e^*_{kl}, \tag{A8}$$

$$\sigma_{ij} \approx T^{(2)}_{ij} \approx \{M_{ijkl} + (N_{kl} - \tfrac{1}{3}M_{ppkl})\delta_{ij}\}e^*_{kl} + \{N_{ij} - \tfrac{1}{3}N_{pp}\delta_{ij} + \chi\delta_{ij}\}e_{rr},$$

after use of Equations (10) and (11). Agreement with (A6) then requires that $\bar{\chi} = \chi$ and $S_{iikl} = 3N_{kl}$, while consistency with (A7) arising from classical elasticity requires the further conditions

$$N_{pp} = 0, \quad M_{ijkl} = S_{ijkl}. \tag{A9}$$

Thus

$$\frac{\partial^2 W^*}{\partial E^*_{ij}\partial E^*_{kl}}(\mathbf{0},1) = C_{ijkl} - \chi\delta_{ij}\delta_{kl}, \tag{A10}$$

with

$$\sum_{k=1}^{3}\frac{\partial^2 W^*}{\partial E^*_{kk}\partial E^*_{ij}}(\mathbf{0},1) = 3\frac{\partial^2 W^*}{\partial J\partial E^*_{ij}}(\mathbf{0},1), \qquad \sum_{p=1}^{3}\frac{\partial W^*}{\partial J\partial E^*_{pp}}(\mathbf{0},1) = 0. \tag{A11}$$

Assuming that the unstrained state is stress-free, the stress-strain relation corresponding to strain energy function $W^*(\boldsymbol{E}^*, J)$ then has the linearisation

$$\sigma_{rs} = \frac{\partial^2 W^*}{\partial E^*_{rs}\partial E^*_{pq}}(\mathbf{0},1)e^*_{pq} + \frac{\partial^2 W^*}{\partial J\partial E^*_{rs}}(\mathbf{0},1)e_{kk} + \chi e_{kk}\delta_{rs}. \tag{A12}$$

(This decomposition (A5) of C_{ijkl}, in which the bulk modulus appears explicitly, seems not to have appeared earlier in the literature).

Appendix B

For some specific forms of $W(\boldsymbol{E})$ previously proposed, the equivalent expression $W^*(\boldsymbol{E}^*, J)$ and consequent forms for $(17)_2$ relating p, J and $\boldsymbol{E}^*$ are given below.

Using the decomposition

$$W^*(\boldsymbol{E}^*, J) = \phi(\lambda^*_1, \lambda^*_2, \lambda^*_3, J) + \chi h(J) \tag{B1}$$

for isotropic materials, Ogden [22] takes

$$\phi = \phi_0(\lambda^*_1, \lambda^*_2) + (J-1)\phi_1(\lambda^*_1, \lambda^*_2), \qquad \lambda^*_3 = (\lambda^*_1\lambda^*_2)^{-1} \tag{B2}$$

when dilatation is not restricted to O(η) while, in [35], he uses

$$\phi = \sum_n \frac{\mu_n}{\alpha_n}\{(\lambda_1^{*\alpha_n} + \lambda_2^{*\alpha_n} + \lambda_3^{*\alpha_n})J^{\alpha_n/3} - 3 - \alpha_n \log J\}. \tag{B3}$$

The exponents α_n and terms μ_n are real material constants, chosen to characterise rubberlike materials.

Blatz [29] proposed the form

$$\phi = \tfrac{1}{2}\mu(I_1 - 3) - \mu(J-1) = \tfrac{1}{2}\mu\{(\lambda_1^{*2} + \lambda_2^{*2} + \lambda_3^{*2})J^{2/3} - 3\} - \mu(J-1). \tag{B4}$$

For $h(J)$, Ogden [22, 35] uses

$$h = \tfrac{1}{9}\{\log J + \tfrac{1}{9}(J^{-9} - 1)\} \quad \text{giving} \quad \frac{\partial h}{\partial J} = \frac{J^9 - 1}{9J^{10}}, \tag{B5}$$

while Blatz [29] uses

$$h = \frac{3(J - 1 - \log J)}{2(1+\nu)} \quad \text{giving} \quad \frac{\partial h}{\partial J} = \frac{3(1 - J^{-1})}{2(1+\nu)}. \tag{B6}$$

Consequently, using (B2) and (B5) as in [22] gives

$$-p = \frac{\partial W^*}{\partial J} = \phi_1(\lambda^*_1, \lambda^*_2) + \chi\frac{J^9 - 1}{9J^{10}},$$

so that

$$J - 1 = -\frac{9\eta}{\mu}\frac{J^{10}}{1 + J + J^2 + \cdots + J^8}\{p + \phi_1(\lambda_1^*, \lambda_2^*)\}. \tag{B7}$$

Similarly, using (B3) and (B6) as in [35] yields

$$\begin{aligned} J - 1 \;=\; & -\frac{9\eta}{\mu}\frac{J^{10}}{1 + J + J^2 + \ldots + J^8} \\ & \times \left[p + \frac{1}{3}\sum_n \mu_n \left\{ (\lambda_1^{*\alpha_n} + \lambda_2^{*\alpha_n} + \lambda_3^{*\alpha_n}) J^{-1+\alpha_n/3} - \frac{3}{J} \right\} \right], \end{aligned} \tag{B8}$$

while using (B4) and (B6) as in [29] gives

$$J - 1 = \tfrac{2}{3}(1 + \nu)\eta\{(1 - p/\mu)J - \tfrac{1}{3}I_1\}. \tag{B9}$$

Solution of any of (B7)–(B9) gives the corresponding function

$$J = J_p(\lambda_1^*, \lambda_2^*, p).$$

Equation (B9) has been used for the computations in Section 5.

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